

# An Integral Inequality and the Riccati-Bernoulli Differential Equation

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## Abstract

We apply an integral inequality to obtain a rigorous apriori estimate of the accuracy of the partial sum to the power series solution of the celebrated RICCATI-BERNOULLI differential equation.

## 1 Introduction

The deservedly celebrated RICCATI-BERNOULLI differential equation:

$$\boxed{y' = x^2 + y^2} \quad (1.1)$$

has the general solution ([1]):

$$\boxed{y(x) = x \frac{J_{\frac{3}{4}}\left(\frac{1}{2}x^2\right) - cJ_{-\frac{3}{4}}\left(\frac{1}{2}x^2\right)}{cJ_{\frac{1}{4}}\left(\frac{1}{2}x^2\right) + J_{-\frac{1}{4}}\left(\frac{1}{2}x^2\right)}} \quad (1.2)$$

where  $c$  is an arbitrary constant and where

$$\boxed{J_n(x) := \frac{x^n}{2^n \Gamma(n+1)} \left\{ 1 - \frac{x^2}{2^2 \cdot 1! \cdot (n+1)} + \frac{x^4}{2^4 \cdot 2! \cdot (n+1)(n+2)} - \cdots \right\}} \quad (1.3)$$

is *the Bessel function of the first kind of order  $n$* , where  $n$  is any real number, and  $\Gamma(n+1)$  is the famous *gamma function*. The very interesting history of (1.1) is detailed in WATSON's standard treatise [3].

The equation (1.1) is an example of a simple differential equation whose solutions form a family of transcendental functions which are essentially distinct from the elementary transcendents.

Unfortunately, the general solution (1.2) does not easily lend itself to a rigorous error analysis of its accuracy in a particular interval of the variable.

We will show how a simple application of an ***integral inequality*** allows one to estimate the accuracy of the partial sum of the TAYLOR series expansion of the solution within the latter's interval of convergence.

## 2 Cauchy's theorem

(In this section we follow [2], Chapter IV, section 5). The general ***Cauchy Problem*** is to solve the ordinary differential equation (ODE) initial-value problem:

$$\boxed{y' = f(x, y), \quad y(x_0) := y_0} \quad (2.1)$$

Let  $f(x, y)$  be expanded in the series

$$f(x, y) = \sum_{i,j} A_{ij} (x - x_0)^i (y - y_0)^j \quad (2.2)$$

convergent for

$$|x - x_0| < R_1 \quad |y - y_0| < R_2 \quad (R_1 > 0, \quad R_2 > 0) \quad (2.3)$$

Then, according to CAUCHY's theorem in the theory of differential equations, the problem (2.1) has a solution  $y(x)$  represented by the series

$$\boxed{y(x) = \sum_{k=0}^{\infty} \frac{y^{(k)}(x_0)}{k!} (x - x_0)^k} \quad (2.4)$$

convergent in some neighborhood of the point  $x_0$ .

CAUCHY's theorem allows one also to indicate the neighborhood of the point  $x_0$ , in which the series (2.4) converges. Namely, let  $M$  be a constant such that

$$|f(x, y)| \leq M \quad (2.5)$$

with

$$|x - x_0| \leq r_1 < R_1 \quad |y - y_0| \leq r_2 < R_2 \quad (2.6)$$

where  $R_1$  and  $R_2$  are numbers, defining the region (2.3) of the convergence of the series (2.2), and  $r_1$  and  $r_2$  are some positive numbers. *Then the series (2.4) converges for*

$$|x - x_0| < r \quad (2.7)$$

where

$$\boxed{r := r_1 \left\{ 1 - e^{-\frac{1}{2M} \frac{r_2}{r_1}} \right\}} \quad (2.8)$$

It should be noted that the *true* interval of convergence is usually much larger than (2.8),

### 3 An integral inequality

We use the notation of (2.1):

**Theorem 1.** *Let  $I$  be the interval,  $x_0 \leq x \leq x_1$ , and suppose that for all  $x \in I$ ,  $f(x, y) > 0$ , and that the differential inequality*

$$y' \leq f\{x_1, y(x)\} \quad (3.1)$$

*also holds there. Then, the integral inequality*

$$\boxed{\int_{x_0}^x \left\{ \frac{1}{f[x_1, y(t)]} \frac{dy}{dt} \right\} dt \leq x - x_0} \quad (3.2)$$

*holds for all  $x \in I$ .*

*Proof.* This is a simple consequence of the elementary calculus sufficient condition that a function be *decreasing* in an interval.

Define

$$g(x) := \int_{x_0}^x \left\{ \frac{1}{f[x_1, y(t)]} \frac{dy}{dt} \right\} dt - (x - x_0). \quad (3.3)$$

Taking the derivative and using the fundamental theorem of calculus we obtain

$$g'(x) = \left\{ \frac{1}{f[x_1, y(x)]} \frac{dy}{dx} \right\} - 1 \quad (3.4)$$

But, the inequality (3.1) shows that

$$g'(x) \leq 0 \quad (3.5)$$

for all  $x \in I$ .

□

### 4 The Riccati-Bernoulli Initial Value Problem

(In this section we follow [2], Chapter IV, section 5, but with some important refinements.)

**Problem:** *It is required to find the first 11 terms of the power series expansion of the solution of the initial value problem*

$$\boxed{y' = x^2 + \frac{y^2}{4}, \quad y(0) := -1} \quad (4.1)$$

*and an interval of convergence for it.* If we specialize the general solution (1.2) to this initial value problem we find the following formula for the exact solution:

$$\boxed{y(x) = \frac{x}{16} \cdot \left\{ \frac{\Gamma\left(\frac{3}{4}\right) J_{\frac{3}{4}}\left(\frac{1}{2}x^2\right) - \sqrt{2}\Gamma\left(\frac{1}{4}\right) J_{\frac{-3}{4}}\left(\frac{1}{2}x^2\right)}{\sqrt{2}\Gamma\left(\frac{1}{4}\right) J_{\frac{1}{4}}\left(\frac{1}{2}x^2\right) + \Gamma\left(\frac{3}{4}\right) J_{\frac{-1}{4}}\left(\frac{1}{2}x^2\right)} \right\}} \quad (4.2)$$

Unfortunately, the computation of the power series solution on the basis of the quotient (4.2), although theoretically possible, is computationally formidable.

It is easier to use the equation (4.1) to compute the derivatives of  $y(x)$  at  $x = 0$  directly:

$$y' = x^2 + \frac{y^2}{4} = \frac{1}{4} \quad (4.3)$$

$$y'' = 2x + \frac{1}{2}yy' = -\frac{1}{8} \quad (4.4)$$

$$y''' = 2 + \frac{1}{2}y'^2 + \frac{1}{2}yy'' = \frac{67}{32} \quad (4.5)$$

$$y^{(4)} = \frac{3}{2}y'y'' + \frac{1}{2}yy''' = -\frac{35}{32} \quad (4.6)$$

$$y^{(5)} = \frac{3}{2}y''^2 + 2y'y''' + \frac{1}{2}yy^{(4)} = \frac{207}{128} \quad (4.7)$$

$$y^{(6)} = 5y''y''' + \frac{5}{2}y'y^{(4)} + \frac{1}{2}yy^{(5)} = -\frac{231}{64} \quad (4.8)$$

$$y^{(7)} = 5y'''^2 + \frac{15}{2}y''y^{(4)} + 3y'y^{(5)} + \frac{1}{2}yy^{(6)} = \frac{26585}{1024} \quad (4.9)$$

$$y^{(8)} = \frac{35}{2}y'''y^{(4)} + \frac{21}{2}y''y^{(5)} + \frac{7}{2}y'y^{(6)} + \frac{1}{2}yy^{(7)} = -\frac{119475}{2048} \quad (4.10)$$

$$y^{(9)} = \frac{35}{2}(y^{(4)})^2 + 28y'''y^{(5)} + 14y''y^{(6)} + 4y'y^{(7)} + \frac{1}{2}yy^{(8)} = \frac{725769}{4096} \quad (4.11)$$

$$y^{(10)} = 63y^{(4)}y^{(5)} + 42y'''y^{(6)} + 18y''y^{(7)} + \frac{9}{2}y'y^{(8)} + \frac{1}{2}yy^{(9)} = -\frac{10509885}{16384} \quad (4.12)$$

Therefore, by the formula (2.4), the first 11 terms of the series solution are:

$$y(x) = -1 + \frac{1}{4}x - \frac{1}{16}x^2 + \frac{67}{192}x^3 - \frac{35}{768}x^4 + \frac{69}{5120}x^5 - \frac{77}{15360}x^6 + \frac{5317}{1032192}x^7 \\ - \frac{2655}{1835008}x^8 + \frac{80641}{165150720}x^9 - \frac{77851}{62914560}x^{10} + \dots \quad (4.13)$$

To find an interval of convergence of this series (4.13) we use the CAUCHY theorem. It  $x$  and  $y$  satisfy the inequalities

$$|x| \leq 0.5 \quad |y + 1| \leq 1$$

then we may conclude that

$$|f(x, y)| \leq |0.25[(y + 1) - 1]^2 + x^2| \\ \leq 0.25(|y + 1| + 1)^2 + |x|^2 \\ \leq 1.25$$

Therefore, in the formula (2.8), we may take

$$r_1 := 0.5 \quad r_2 := 1 \quad M := 1.25$$

and the value of  $r$  we obtain is:

$$r = 0.5 (1 - e^{-0.8}) = 0.2753355 \dots \quad (4.14)$$

Therefore, the power series solution (4.13) most certainly converges for  $|x| \leq 0.27$ .

## 5 The Accuracy of a Partial Sum from the Integral Inequality

We will consider the following concrete problem although the principles are of general applicability.

**Problem:** *It is required to determine the accuracy of the partial sum of degree 9 of the power series solution (4.13) in the interval  $0 \leq x \leq 0.2$ .*

Since the series (4.13) is the MACLAURIN expansion of  $y(x)$ , we must estimate the remainder term,  $R_9(x)$ , which we write in the LAGRANGE form:

$$\boxed{R_9(x) := \frac{y^{(10)}(\Theta_9)}{10!} x^{10}} \quad (5.1)$$

where  $0 < \Theta_9 < 0.2$ . We have to estimate  $y^{(10)}(x)$ , the formula for which is given in (4.12), for all values of  $x$  in the interval  $0 \leq x \leq 0.2$ . The formulas (4.3) through (4.11) show us, finally, that *we must estimate  $y(x)$ , itself in  $0 \leq x \leq 0.2$ .*

The estimate of  $y(x)$  via *our integral inequality* (3.2) constitutes the novelty in this paper. Maintaining the notation of (3.2) we see that

$$x_0 := 0 \quad x_1 := 0.2, \quad (5.2)$$

that the right hand side

$$f(x, y) := x^2 + \frac{y^2}{4} > 0 \quad (5.3)$$

on  $I$ , and the differential inequality (3.1) becomes:

$$\frac{dy}{dx} \leq 0.04 + \frac{y(x)^2}{4} \quad (5.4)$$

Therefore, the integral inequality (3.2) becomes

$$\int_0^x \left\{ \frac{1}{0.04 + \frac{y(t)^2}{4}} \frac{dy}{dt} \right\} dt \leq x \quad (5.5)$$

But,

$$\left\{ \frac{1}{0.04 + \frac{y(t)^2}{4}} \frac{dy}{dt} \right\} = \frac{d}{dt} \left\{ 10 \arctan \frac{y(t)}{0.4} \right\} \quad (5.6)$$

Therefore,

$$10 \arctan \frac{y(x)}{0.4} - 10 \arctan \frac{-1}{0.4} \leq x \quad (5.7)$$

or

$$\arctan \frac{y(x)}{0.4} \leq \frac{x}{10} + \arctan \frac{-1}{0.4} \quad (5.8)$$

and taking the tangent of both sides and reducing we obtain the estimate

$$\boxed{y(x) \leq \frac{\frac{2}{5} \tan\left(\frac{x}{10}\right) - 1}{1 + \frac{x}{4}}} \quad (5.9)$$

which holds for all  $x \in I$ ,

The function on the right-hand side of (5.9) is monotonically increasing in  $I$ , and

$$\frac{\frac{2}{5} \tan\left(\frac{0.2}{10}\right) - 1}{1 + \frac{0.2}{4}} = -0.9447608 \dots < -0.94$$

and we have proved that the following inequality is true for all  $x \in I$ :

$$\boxed{-1 \leq y(x) \leq -0.94.} \quad (5.10)$$

(Note: the true value of  $y(0.2)$  is

$$y(0.2) = -0.9497771 \dots,$$

so the estimate (5.9), with an error of  $-0.00501 \dots$ , or about 0.53%, is quite good!).

Now we must estimate  $y'(x), \dots, y^{(10)}(x)$  using the formulas (4.3) through (4.12).

**Theorem 2.** *The following inequalities are valid for all  $x \in I$ .*

$$\begin{aligned} -1 &\leq y(x) \leq -0.94, \\ 0.22 &\leq y'(x) \leq 0.29, \\ -0.15 &\leq y''(x) \leq 0.3, \\ 1.87 &\leq y'''(x) \leq 2.12 \\ -1.13 &\leq y^{(4)}(x) \leq -0.74, \\ 1.17 &\leq y^{(5)}(x) \leq 1.93, \\ -3.38 &\leq y^{(6)}(x) \leq 2.23, \\ 14.59 &\leq y^{(7)}(x) \leq 27.12, \\ -61.96 &\leq y^{(8)}(x) \leq -22.73, \\ 92.03 &\leq y^{(9)}(x) \leq 146.76, \end{aligned}$$

and finally

$$\boxed{-665.9 < y^{(10)}(x) < 281.} \quad (5.11)$$

*Proof.* All of our estimates come from worst case values applied to each of the summands in the formulas.

By (4.3)

$$\boxed{y' = x^2 + \frac{y^2}{4}.} \quad (5.12)$$

Therefore, using (5.10) and (5.12), we conclude that for all  $x \in I$ ,

$$y' < 0.2^2 + \frac{(-1)^2}{4} = 0.29,$$

while

$$y' > 0^2 + \frac{(-.94)^2}{4} = .2209 > .22.$$

Therefore, we obtain the bounds

$$\boxed{.22 < y'(x) < .29.} \quad (5.13)$$

By (4.4)

$$\boxed{y'' = 2x + \frac{1}{2}yy'.} \quad (5.14)$$

Therefore, using (5.13) and (5.14), we conclude that for all  $x \in I$ ,

$$y'' < 2(0.02) + \frac{1}{2}(-0.94)(.22) = 0.2966 < .3,$$

while

$$y'' > 2(0) + \frac{1}{2}(-1)(.29) = -0.145 > -0.15.$$

Therefore, we obtain the bounds

$$\boxed{-0.15 < y''(x) < 0.3.} \quad (5.15)$$

By (4.5)

$$\boxed{y''' = 2 + \frac{1}{2}y'^2 + \frac{1}{2}yy''.} \quad (5.16)$$

Therefore, using (5.13) , (5.15) and (5.16), we conclude that for all  $x \in I$ ,

$$y''' < 2 + \frac{1}{2}(0.29)^2 + \frac{1}{2}(-0.15)(-1) = 2.11705 < 2.12,$$

while

$$y''' > 2 + \frac{1}{2}(0.22)^2 + \frac{1}{2}(0.3)(-1) = 1.8742 > 1.87.$$

Therefore, we obtain the bounds

$$\boxed{1.87 < y'''(x) < 2.12.} \quad (5.17)$$

By (4.6)

$$\boxed{y^{(4)} = \frac{3}{2}y'y'' + \frac{1}{2}yy'''.} \quad (5.18)$$

Therefore, using (5.13) , (5.15), (5.17), and (5.18), we conclude that for all  $x \in I$ ,

$$y^{(4)} < \frac{3}{2}(0.29)(0.3) + \frac{1}{2}(-0.94)(1.87) = -0.7484 < -0.74,$$

while

$$y^{(4)} > \frac{3}{2}(0.29)(-0.15) + \frac{1}{2}(-1)(2.12) = -1.12525 > -1.13.$$

Therefore, we obtain the bounds

$$\boxed{-1.13 < y^{(4)}(x) < -0.74.} \quad (5.19)$$

By (4.7)

$$\boxed{y^{(5)} = \frac{3}{2}y''^2 + 2y'y''' + \frac{1}{2}yy^{(4)}.} \quad (5.20)$$

Therefore, using (5.13) , (5.15), (5.17), (5.19) and (5.20), we conclude that for all  $x \in I$ ,

$$y^{(5)} < \frac{3}{2}(0.3)^2 + 2(0.29)(2.12) + \frac{1}{2}(-1)(-1.13) = 1.9296 < 1.93,$$

while

$$y^{(5)} > \frac{3}{2}(0)^2 + 2(0.22)(1.87) + \frac{1}{2}(-0.94)(-0.74) = 1.1706 > 1.17.$$

Therefore, we obtain the bounds

$$\boxed{1.17 < y^{(5)}(x) < 1.93.} \quad (5.21)$$

By (4.8)

$$\boxed{y^{(6)} = 5y''y''' + \frac{5}{2}y'y^{(4)} + \frac{1}{2}yy^{(5)}.} \quad (5.22)$$

Therefore, using (5.13) , (5.15), (5.17), (5.19), (5.21) and (5.22), we conclude that for all  $x \in I$ ,

$$y^{(6)} < 5(.3)(2.12) + \frac{5}{2}(.22)(-0.74) + \frac{1}{2}(-0.94)(1.17) = 2.2231 < 2.23,$$

while

$$y^{(6)} > 5(-0.15)(2.12) + \frac{5}{2}(.29)(-1.13) + \frac{1}{2}(-1)(1.93) = -3.37425 > -3.38.$$

Therefore, we obtain the bounds

$$\boxed{-3.38 < y^{(6)}(x) < 2.23.} \quad (5.23)$$



By (4.9)

$$\boxed{y^{(7)} = 5y'''^2 + \frac{15}{2}y''y^{(4)} + 3y'y^{(5)} + \frac{1}{2}yy^{(6)}}. \quad (5.24)$$

Therefore, using (5.13) , (5.15), (5.17), (5.19), (5.21), (5.23) and (5.24), we conclude that for all  $x \in I$ ,

$$y^{(7)} < 5(2.12)^2 + \frac{15}{2}(-0.15)(-1.13) + 3(.29)(1.93) + \frac{1}{2}(-1)(-3.38),$$

which

$$= 27.11235 < 27.12,$$

while

$$y^{(7)} > 5(1.87)^2 + \frac{15}{2}(0.3)(-1.13) + 3(.22)(1.17) + \frac{1}{2}(-1)(2.23),$$

which

$$= 14.5992 > 14.59.$$

Therefore, we obtain the bounds

$$\boxed{14.59 < y^{(7)}(x) < 27.12}. \quad (5.25)$$

By (4.10)

$$\boxed{y^{(8)} = \frac{35}{2}y'''y^{(4)} + \frac{21}{2}y''y^{(5)} + \frac{7}{2}y'y^{(6)} + \frac{1}{2}yy^{(7)}}. \quad (5.26)$$

Therefore, using (5.13) , (5.15), (5.17), (5.19), (5.21), (5.23), (5.25)and (5.26), we conclude that for all  $x \in I$ ,

$$y^{(8)} < \frac{35}{2}(1.87)(-0.74) + \frac{21}{2}(0.3)(1.93) + \frac{7}{2}(.29)(2.23) + \frac{1}{2}(-0.94)(14.59),$$

which

$$= -22.73085 < -22.73,$$

while

$$y^{(8)} > \frac{35}{2}(2.12)(-1.13) + \frac{21}{2}(-0.15)(1.93) + \frac{7}{2}(.29)(-3.38) + \frac{1}{2}(-1)(27.12),$$

which

$$= -61.95345 > -61.96.$$

Therefore, we obtain the bounds

$$\boxed{-61.96 < y^{(8)}(x) < -22.73}. \quad (5.27)$$

By (4.11)

$$\boxed{y^{(9)} = \frac{35}{2}(y^{(4)})^2 + 28y'''y^{(5)} + 14y''y^{(6)} + 4y'y^{(7)} + \frac{1}{2}yy^{(8)}}. \quad (5.28)$$

Therefore, using (5.13) , (5.15), (5.17), (5.19), (5.21), (5.23), (5.25), (5.27) and (5.28), we conclude that for all  $x \in I$ ,

$$y^{(9)} < \frac{35}{2}(-1.13)^2 + 28(2.12)(1.93) + 14(0.3)(2.23) + 4(.29)(27.12) + \frac{1}{2}(-1)(-61.96),$$

which

$$= 146.75575 < 146.76,$$

while

$$y^{(9)} > \frac{35}{2}(-0.74)^2 + 28(1.87)(1.17) + 14(-0.15)(2.23) + 4(.22)(14.59) + \frac{1}{2}(-0.94)(-27.73),$$

which

$$= 92.0335 > 92.03.$$

Therefore, we obtain the bounds

$$\boxed{92.03 < y^{(9)}(x) < 146.76}. \quad (5.29)$$

By (4.12)

$$\boxed{y^{(10)} = 63y^{(4)}y^{(5)} + 42y'''y^{(6)} + 18y''y^{(7)} + \frac{9}{2}y'y^{(8)} + \frac{1}{2}yy^{(9)}}. \quad (5.30)$$

Therefore, using (5.13) , (5.15), (5.17), (5.19), (5.21), (5.23), (5.25), (5.27), (5.29) and (5.30), we conclude that for all  $x \in I$ ,

$$y^{(10)} < 63(-0.74)(1.17) + 42(2.12)(2.23) + 18(0.3)(27.12) + \frac{9}{2}(0.22)(-22.73) + \frac{1}{2}(-0.94)(92.03),$$

which

$$= 280.9922 < 281,$$

while

$$y^{(10)} > 63(-1.13)(1.93) + 42(2.12)(-3.38) + 18(-0.15)(27.12) + \frac{9}{2}(0.29)(-61.96) + \frac{1}{2}(-1)(146.76),$$

which

$$= -665.8137 > -665.9.$$

Therefore, we obtain the bounds

$$\boxed{-665.9 < y^{(10)}(x) < 281}. \quad (5.31)$$

*The inequality (5.31) was the goal of this long and detailed computation!*

□

Now we can state the accuracy of the partial sum:

**Theorem 3.** *For all  $x$  in  $0 \leq x \leq 0.2$ , the partial sum of degree 9:*

$$y(x) \approx -1 + \frac{1}{4}x - \frac{1}{16}x^2 + \frac{67}{192}x^3 - \frac{35}{768}x^4 + \frac{69}{5120}x^5 - \frac{77}{15360}x^6 + \frac{5317}{1032192}x^7 - \frac{2655}{1835008}x^8 + \frac{80641}{165150720}x^9 \quad (5.32)$$

*approximates the true value,  $y(x)$ , of the solution series, (4.13), with an error that does not exceed 2 units in the eleventh decimal place.*

*Proof.* The estimate

$$|R_9(x)| = \left| \frac{y^{(10)}(\Theta_9)}{10!} x^{10} \right| \leq \frac{665.9}{10!} (0.2)^{10} = 1.878 \dots 10^{-11} < 2 \cdot 10^{-11}$$

completes the proof. □

## 6 Conclusions

Our integral inequality (3.2) can be applied to wide classes of differential equations.

For example, our method allows us to prove that in the interval  $0 \leq x \leq 0.4$  the polynomial

$$\bar{y}(x) := 1 + \frac{x}{4} + \frac{3}{16}x^2 + \frac{7}{192}x^3 + \frac{1}{96}x^4 + \frac{1}{200}x^5$$

approximates the true solution,  $y(x)$ , of the initial value problem

$$4y' = x + y^2, \quad y(0) := 1$$

with an error that does not exceed 2 units in the fifth decimal place.

We chose the RICCATI-BERNOULLI equation because it illustrates the process so perfectly and because a direct estimate of the accuracy of the partial sum of the series solution is troublesome.

We did not investigate the accuracy for the *negative* half of the interval, i.e., for  $-0.2 \leq x \leq 0$ , which we leave as an exercise for the reader. The only change occurs in the estimate of  $y''(x)$  since then  $x$  can be equal to negative numbers.

Finally we observe that we did not exploit the *sign* of the error. In fact, our estimate (5.31) allows us to say that the error we commit is between an error in *defect* smaller than 2 units in the eleventh decimal place and an error in *excess* smaller than 8 units in the twelfth decimal place. Therefore we can *centralize the error* by adding the term

$$\frac{281 - 665.9}{2 \cdot 10!} x^{10} = \frac{-1283}{24192000} x^{10}$$

to our polynomial (5.32) and obtain an approximating polynomial whose maximum error is between  $\pm 1.4$  units in the 11th decimal place.

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## References

- [1] C.H. Edwards, D.E. Penny, *Differential Equations and Boundary Value Problems*, Prentice Hall, Upper Saddle River, New Jersey, 2000.
- [2] I.P. Mysovskih, *Lectures on Numerical Methods*, Wolters-Noordhoff, Gronigen, The Netherlands, 1969.
- [3] G.N. Watson *A Treatise on the Theory of Bessel Functions*, Cambridge University Press, New York, New York, 1958.